Fusion rules and conditional statistics in turbulent advection

Emily S. C. Ching,[1] Victor S. L'vov,[2] and Itamar Procaccia[2]
[1]*Department of Physics, the Chinese University of Hong Kong, Shatin, Hong Kong*
[2]*Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*
(Received 29 April 1996)

Fusion rules in turbulence address the asymptotic properties of many-point correlation functions when some of the coordinates are very close to each other. Here we put to the experimental test some nontrivial consequences of the fusion rules for scalar correlations in turbulence. To this aim we examine passive turbulent advection as well as convective turbulence. Adding one assumption to the fusion rules, one obtains a prediction for universal conditional statistics of gradient fields. We examine the conditional average of the scalar dissipation field $\langle \nabla^2 T(\mathbf{r}) | T(\mathbf{r}+\mathbf{R}) - T(\mathbf{r}) \rangle$ for R in the inertial range and find that it is linear in $T(\mathbf{r}+\mathbf{R})-T(\mathbf{r})$ with a fully determined proportionality constant. The implications of these findings for the general scaling theory of scalar turbulence are discussed. [S1063-651X(96)50311-9]

PACS number(s): 47.27.Gs, 47.27.Jv, 05.40.+j

The aim of this paper is to present an analysis of experimental data [1–3] pertaining to turbulent scalar advection and to discuss the implications of this analysis in the context of fusion rules and conditional averages. We begin with a short theoretical background of the issues in order to make this paper self-contained. Turbulent advection is described mathematically by the equation of motion for a scalar field $T(\{\mathbf{r}\},t)$,

$$[\partial_t + \mathbf{u}(\mathbf{r},t)\cdot \boldsymbol{\nabla}] T(\mathbf{r},t) = \kappa \nabla^2 T(\mathbf{r},t), \tag{1}$$

where κ is the scalar diffusivity and $\mathbf{u}(\mathbf{r},t)$ is the turbulent velocity field responsible for the advection of $T(\mathbf{r},t)$. The problem of ''passive'' scalar advection is the one in which the properties of $\mathbf{u}(\mathbf{r},t)$ are not affected by those of the scalar $T(\mathbf{r},t)$. In ''active'' scalar problems, like turbulent convection, the velocity field and its statistical properties are coupled with those of the scalar field and Eq. (1) has to be supplemented with an additional equation for $\mathbf{u}(\mathbf{r},t)$ and $T(\mathbf{r},t)$. In our thinking below we consider passive as well as active scalar fields. In both cases we are interested in the limit of large Péclet number Pe, which is defined as $U_L L/\kappa$, where U_L is the typical velocity difference across the outer scale L of turbulence.

The statistical properties of the scalar fields are commonly discussed in terms of the so-called structure functions $S_{2n}(\mathbf{R})$ defined as

$$S_{2n}(\mathbf{R}) \equiv \langle [\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})]^{2n} \rangle, \tag{2}$$

where $\langle \cdots \rangle$ stands for an ensemble average and we denote $\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R}) \equiv T(\mathbf{r}+\mathbf{R},t) - T(\mathbf{r},t)$, In writing this equation, we assume that the statistics of the velocity field leads to a stationary and spatially homogeneous ensemble of the scalar T. If the statistics are also isotropic, then S_{2n} becomes a function of R only, independent of the direction of $\mathbf{R}$. The scaling exponents of the structure functions $S_{2n}(R)$ characterize their R dependence in the limit of large Pe,

$$S_{2n}(R) \propto R^{\zeta_{2n}}, \tag{3}$$

when R is in the ''inertial'' interval of scales that will be discussed later in this paper. One of the fundamental questions in the theory of turbulent advection is what the numerical values of the exponents ζ_{2n} are and whether they conform to classical Kolmogorov-type arguments or, rather, exhibit the phenomenon of multiscaling.

An important equation to analyze in this context is the so-called balance equation, which is obtained by writing Eq. (1) twice at points $\mathbf{r}$ and $\mathbf{r}+\mathbf{R}$, subtracting the equations, and multiplying the result by $2n[T(\mathbf{r}+\mathbf{R},t) - T(\mathbf{r},t)]^{2n-1}$. Taking the ensemble average and using the symmetry between the two points analyzed, one finds the balance equation

$$D_{2n}(R) = J_{2n}(R), \tag{4}$$

where $D_{2n}(R)$ stems from the convective term in (1) and $J_{2n}(R)$ stems from the diffusion term

$$J_{2n}(R) = -4n\kappa \langle \nabla^2 T(\mathbf{r}) [\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})]^{2n-1} \rangle. \tag{5}$$

It was argued recently, by Kraichnan [4] and later in Refs. [5–7] that balance equations play a very important role in providing nonperturbative relations that can determine, or severely constrain, the values of the scaling exponents ζ_{2n}. A good example is Kraichnan's model of passive scalar advection [8], in which the velocity field $\mathbf{u}$ is δ correlated in time, but exhibits power-law scaling in space. In this case the convective term D_{2n} can be calculated exactly in terms of S_{2n} [4],

$$D_{2n}(R) = -R^{1-d} \frac{\partial}{\partial R} R^{d-1} h(R) \frac{\partial}{\partial R} S_{2n}(R), \tag{6}$$

with d being the space dimension and $h(R)$ the scalar part of the eddy diffusivity $h(R) \propto R^{\zeta_h}$, with ζ_h a scaling exponent. If we could represent exactly also the right-hand side (RHS) $J_{2n}(R)$ in terms of $S_{2n}(R)$, we could evaluate all the scaling exponents ζ_{2n} from the balance equation (4). Here is where the fusion rules come in. The fusion rules appear naturally in the analytic theory of Navier-Stokes turbulence [7,9–11] and passive-scalar turbulent advection [6,11,12] and determine the analytic structure of n-point correlation functions when a group of coordinates tend towards each other. In the case of scalar advection we consider simultaneous many-point correlation functions of field differences

$$\mathcal{F}_{2n}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2n}) = \langle \Delta T(\mathbf{r}_0,\mathbf{r}_1)\Delta T(\mathbf{r}_0,\mathbf{r}_2)\cdots\Delta T(\mathbf{r}_0,\mathbf{r}_n)\rangle. \tag{7}$$

We note that the previously defined structure functions S_{2n} are obtained by ''fusing'' all the coordinates $\mathbf{r}_1,\ldots,\mathbf{r}_{2n}$ to one coordinate $\mathbf{r}_0+\mathbf{R}$. The fusion rules were derived in [11] for systems that enjoy universality of the scaling exponents (i.e., the scaling exponents do not depend on the detailed form of the driving of the turbulent flows) and whose correlation functions $\mathcal{F}_{2n}$ are homogeneous functions of their arguments,

$$\mathcal{F}_{2n}(\lambda\mathbf{r}_0|\lambda\mathbf{r}_1,\ldots,\lambda\mathbf{r}_{2n}) = \lambda^{\zeta_{2n}}\mathcal{F}_{2n}(\mathbf{r}_0|\mathbf{r}_1,\ldots,\mathbf{r}_{2n}). \tag{8}$$

This form applies whenever all the distances $|\mathbf{r}_i-\mathbf{r}_0|$ are in the so-called inertial range, between the outer scale L and the appropriate dissipative scale of the system, denoted below as η. The fusion rules address the asymptotic properties of $\mathcal{F}_{2n}$ when a group of p points, $p<2n-1$, tend towards $\mathbf{r}_0$ ($|\mathbf{r}_i-\mathbf{r}_0|\sim\rho$ for all $i\leqslant p$), while all the other coordinates remain at a larger distance R from $\mathbf{r}_0$ ($|\mathbf{r}_i-\mathbf{r}_0|\sim R$ for $i>p$ and $R\gg\rho$). In particular, under the two general assumptions of scale invariance and universality of the scaling exponents the fusion rules state that to leading order in ρ/R,

$$\mathcal{F}_{2n}(\mathbf{r}_0|\mathbf{r}_0+\boldsymbol{\rho},\mathbf{r}_0+\mathbf{R},\ldots,\mathbf{r}_0+\mathbf{R}) \sim \frac{S_2(\rho)}{S_2(R)}S_{2n}(R). \tag{9}$$

This forms holds as long as ρ is in the inertial range.

We show now how to use this fusion rule to calculate $J_{2n}(R)$. First write it as

$$J_{2n}(R) = -4\kappa n \lim_{\rho\to 0} \nabla^2_\rho \mathcal{F}_{2n}(\mathbf{r}_0|\mathbf{r}_0+\boldsymbol{\rho},\mathbf{r}_0+\mathbf{R},\ldots,\mathbf{r}_0+\mathbf{R}).$$

In using the fusion rule (9) to evaluate this quantity we interpret the limit $\rho\to 0$ as a limit $\rho\to\eta$. This seems natural for large Péclet numbers when $\eta\to 0$. It is important, however, to stress that there is a hidden assumption here. We expect the function $\mathcal{F}_{2n}(\boldsymbol{r}_0|\boldsymbol{r}_0+\boldsymbol{\rho},\boldsymbol{r}_0+\mathbf{R},\ldots,\mathbf{r}_0+\mathbf{R})$, which is a function of $\boldsymbol{\rho}$ and $\mathbf{R}$, to change its analytic behavior as a function of ρ. This change occurs at the viscous crossover scale η. The issue is whether this crossover scale is n and R independent. That this is so has been *proven* for Kraichnan's model of turbulent advection [11], but not in general. We believe that this is more generally true due to the linearity of the equation of motion (1), independently of the statistical properties of the driving velocity field. The experimental results that we discuss later in this paper will strongly indicate that this is the case in a wide context of scalar turbulent fields. We caution the reader that this is not so in Navier-Stokes turbulence. With this in mind we write

$$J_{2n}(R) \sim -4\kappa n[\nabla^2_\rho S_2(\rho)|_{\rho=\eta}]S_{2n}(R)/S_2(R). \tag{10}$$

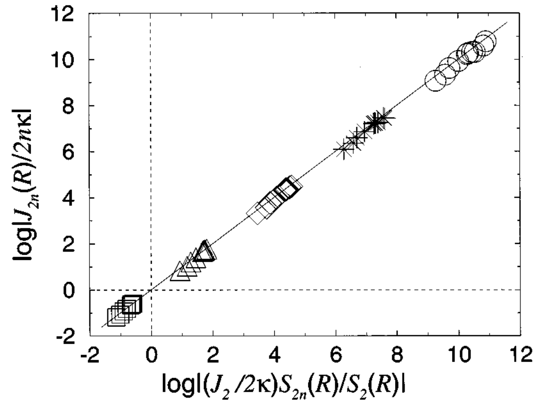

FIG. 1. Plot of $\log_e|J_{2n}(R)/(2n\kappa)|$ vs $\log_e|(2\kappa)^{-1}J_2S_{2n}(R)/S_2(R)|$ for $n=2$ (squares), 3 (triangles), 4 (diamonds), 5 (stars), and 6 (circles) and R in the inertial range. The data are taken from Ref. [1]. The line is not a fit, but the theoretical expectation with slope 1 and intercept 0.

Using the fact that the mean of the scalar dissipation field, denoted $\overline{\epsilon}$, is evaluated as $\overline{\epsilon}\sim\kappa[\nabla^2_\rho S_2(\rho)|_{\rho=\eta}]$ and also the fact that in the inertial range $J_2(R)=-4\overline{\epsilon}$, we write

$$J_{2n}(R) = nC_{2n}J_2S_{2n}(R)/S_2(R), \tag{11}$$

where C_{2n} is an as yet unknown dimensionless coefficient, but $C_2=1$. Equation (11) was suggested for Kraichnan's model in [4] and derived in [6]. Here we propose that it holds in a much wider context. To this end we turn now to the analyses of experimental data.

We first display experimental results that confirm the theoretical prediction (11). The results show that to a good accuracy $C_{2n}\approx 1$ for all n and R. The theoretical consequences of this n and R independence of C_{2n} will be discussed after examining the data.

We use temperature data measured in the wake of a heated cylinder [1]. Water of speed 5 m/s flowed past a heated cylinder of diameter 19 mm (Reynolds number equal to 9.5×10^4), The temperature was measured at a fixed point downstream of the cylinder on the wake center line. The cylinder was heated so slightly that the buoyancy term was unimportant and temperature acted as a passive scalar. Temperature was measured as a function of time, and we use here the standard Taylor hypothesis that surrogates time derivatives for space derivatives. In Fig. 1 we display $J_{2n}(R)/2n\kappa$ as a function of $(2\kappa)^{-1}J_2S_{2n}(R)/S_2(R)$ for n varying from 2 to 6 and for various R values in the inertial range. We see that all the points fall on a line whose slope is unity to high accuracy and whose intercept (in log-log plot) is very closely zero. This good agreement is a confirmation of the validity of the fusion rules. In addition, this agreement lends support to the *assumption* that η is n and R independent. It should be stressed that individual tests at various values of n as a function of R corroborate the same conclusion, i.e., Eq. (11) is supported by the experimental data with C_{2n} being near unity. The most sensitive test of the alleged constancy of the coefficients C_{2n} is obtained by dividing $J_{2n}(R)$ by $nJ_2S_{2n}(R)/S_2(R)$ for all the available values n and R. The result of such a test is shown in Fig. 2. We see that all the measured values of C_{2n} are concentrated within

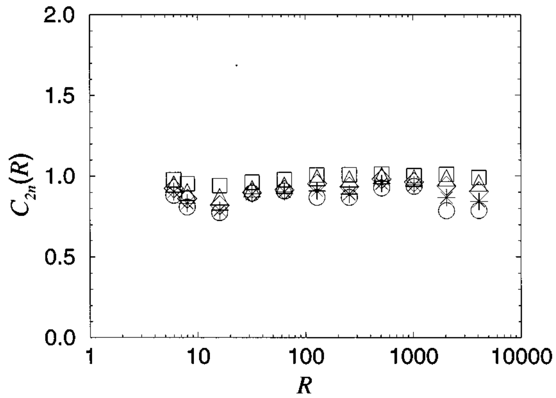

FIG. 2. Detailed test of the coefficient C_{2n}; see the text for details. The symbols are the same as in Fig. 1. The small systematic decrease of C_{2n} with n may be due to insufficient accuracy at the tails of the probability distribution that become more important at large values of n. The separation R was measured in units of sampling time, using the Talor hypothesis. The inertial interval is approximately $10<R<1000$.

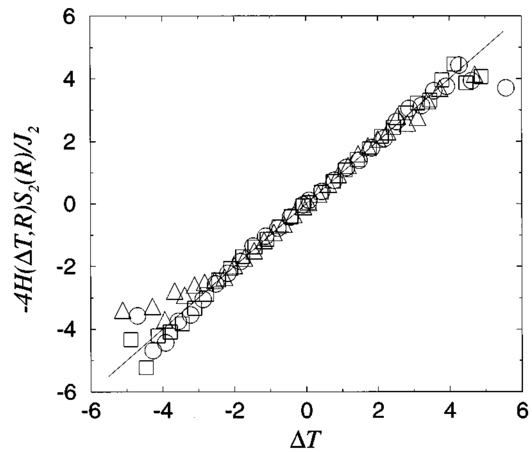

FIG. 3. Conditional average in Eq. (13) as measured from Ref. [1] normalized by the measured value of $-J_2/4S_2(R)$ as a function of $\Delta T(R)$ for three different values of R measured in units of the sampling time. The different R values are designated by triangles ($R=16$), squares ($R=128$), and circles ($R=1024$), respectively.

the interval (0.75,1) for all separation within the inertial interval. Considering the fact that the quantities themselves vary in this region over five orders of magnitude, we interpret this as a good indication of the independence of C_{2n} of R and n. The R independence is very clear and is a direct test of the fusion rules. The weak n dependence seems to indicate that C_{2n} decreases slightly with n; this may arise from the limited accuracy of the data. We are reluctant to make a strong claim about the accuracy of 10th- or 12th-order structure functions.

Let us accept for now the evidence that the coefficients C_{2n} in Eq. (11) are n independent and look for a way to understand it. Note that $J_{2n}(R)$ can be written exactly in terms of conditional averages in the form

$$J_{2n}(R)=-4n\kappa\int d\delta\, T(\mathbf{r},\mathbf{r}+\mathbf{R})P[\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})]\times[\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})]^{2n-1}\langle\nabla^2T(\mathbf{r})|\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})\rangle. \tag{12}$$

We see that the conditional average $\langle\nabla^2T(\mathbf{r})|\delta T(\mathbf{r},\mathbf{r}+\mathbf{R})]\rangle$ appears as a natural object that needs to be determined. If we make the assumption that the conditional average, which in general is a function of the two variables R and ΔT, is factorizable as a function of R times a function of ΔT, then the only possible such form is

$$-4\kappa\langle\nabla^2T(\mathbf{r})|\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})\rangle=\frac{J_2}{S_2(R)}\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R}). \tag{13}$$

With this form in (12) we regain the RHS of (11), but since the conditional average cannot be a function of n, the coefficient C_{2n} *must* be n independent. We note that Kraichnan conjectured that the conditional average is *linear* in $\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})$ in the context of the Kraichnan model and numerical simulations supporting this conjecture were presented [5]. Moreover, linearity approximations of conditional average of the form $\langle\nabla^2X|X\rangle$ and relations such as (11) have also been studied earlier by Ching [13] and by Pope and Ching [14]. In this paper we propose that the linearity of the conditional average in $\delta T(\mathbf{r},\mathbf{r}+\mathbf{R})$ is a general property of a wider variety of turbulent advection problems. From our discussion, it is clear that other results on conditional statistics can be derived in a similar fashion.

Before we proceed to the implications of (13), we present the experimental evidence of its validity. In Fig. 3 we present results from the same data set that was used above. We show the conditional average as a function of $\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})$ for various values of R. The line passing through the data points is not a fit, but rather the line required by Eq. (13). We note that points belonging to different values of R fall on the same line, indicating that indeed the conditional average is a function of $\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})$ times a function of R, and that we identified correctly the function of R as $J_2/S_2(R)$. To test the generality of this result we analyzed a second data set from the convective hard turbulence regime of the well-documented experiment by Libchaber and co-workers [2,3]. The experiment was performed in a cylindrical box of helium gas heated from below and the Rayleigh number can be as high as 10^{15}. The box has a diameter of 20 cm and a height of 40 cm. The temperature at the center of the box was measured as a function of time and we use the same Taylor hypothesis to analyze the conditional average. The results are shown in Fig. 4. Although we see larger statistical scatter at the ends of the plot, the basic assertion of linearity with the correct slope is confirmed.

As explained, the linearity of the conditional average in ΔT [Eq. (13)] was not derived from first principles. To stress the theoretical interest in such a derivation we consider briefly another form of $J_{2n}(R)$ that is obtained by moving around one of the gradients in (5). Up to a term that is negligible for R in the inertial range we can write

$$J_{2n}(R)=-4n(2n-1)\kappa\langle|\nabla T(\mathbf{r})|^2[\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})]^{2n-2}\rangle.$$

Accepting Eq. (11) with $C_{2n}=1$ we can write

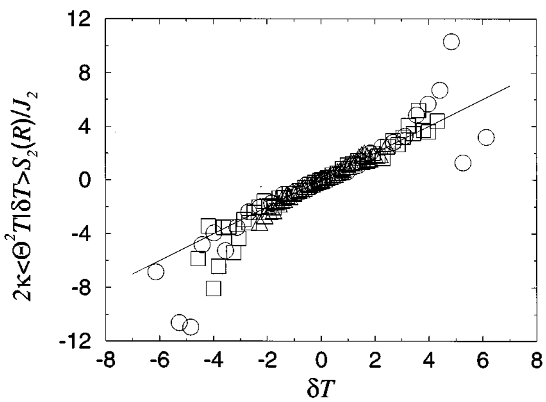

FIG. 4. Same as Fig. 3 but computed from Refs. [2,3].

$$-4\kappa\langle|\nabla T(\mathbf{r})|^2[\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})]^{2n-2}\rangle = J_2 S_{2n}(R)/(2n-1)S_2(R). \qquad (14)$$

The LHS can be written, similarly to (12), in terms of the conditional average $\langle|\nabla T(\mathbf{r})|^2|\Delta T(\mathbf{r},\mathbf{r}+\mathbf{R})\rangle$. It is obvious, however, that now we *cannot* assume that this quantity factorizes into a function of ΔT times a function of R. If it did, the dependence on ΔT must have been $(\Delta T)^2$ in order to give us $S_{2n}(R)$ on the RHS of (14). But we can never obtain in this way the explicit $1/(2n-1)$ factor. This underlines the fact that the factorization in (13) is far from being obvious or trivial. Currently, we do not know the deep reason why the conditional averages of $\nabla^2 T$ afford factorization. We pose this as an important issue for further theoretical research.

Finally, we comment on the implications of these findings for the exponents ζ_n. As discussed above, if we know the functional form of J_{2n} and the coefficient, we can use the balance equation (4) to compute the scaling exponents, provided that we know the nonlinear term D_{2n}. In the context of the Kraichnan model the latter is known exactly, and the balance equation leads to a quadratic equation for the exponents ζ_n, with the solution

$$\zeta_{2n}=\tfrac{1}{2}[\zeta_2-d+\sqrt{(\zeta_2+d)^2+4d\zeta_2(n-1)}]. \qquad (15)$$

These are the exponents that were conjectured by Kraichnan. We note that these exponents are in disagreement with the calculations of Refs. [12,15], which attempted to compute the exponents by perturbative methods using as a small parameter either $2-\zeta_2$ or the inverse dimension $1/d$. If it turns out indeed that (15) is the correct nonperturbative result, one needs to carefully rethink the meaning of these perturbative calculations.

I.P. is grateful for the financial support of the Chinese University of Hong Kong that made our collaboration possible. We thank K.R. Sreenivasan and A. Libchaber for their experimental data. This work was supported in part by the German-Israeli Foundation, the Minerva Center for Nonlinear Physics, and the Naftali and Anna Backenroth-Bronicki Fund for Research in Chaos and Complexity. The work of E.S.C.C. was also supported in part by the Hong Kong Research Grants Council (Grant No. 458/95P).

[1] The data were obtained from K.R. Sreenivasan.
[2] F. Heslot, B. Castaing, and A. Libchaber, Phys. Rev. A **36**, 5870 (1987).
[3] M. Sano, X.-Z Wu, and A. Libchaber, Phys. Rev A **40**, 6421 (1989).
[4] R.H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
[5] R.H. Kraichnan, V. Yakhot, and S. Chen, Phys. Rev. Lett. **75**, 240 (1995)
[6] A.L. Fairhall, O. Gat, V.S. L'vov, and I. Procaccia, Phys. Rev. E **53**, 3518 (1996).
[7] V.S. L'vov and I. Procaccia, Phys. Rev. E **53**, 3468 (1996).
[8] R.H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[9] V.V. Lebedev and V.S. L'vov, Pis'ma Zh. Éksp. Teor. Fiz. **59**, 546 (1994) [JETP Lett. **59**, 577 (1994)].
[10] V.S. L'vov and I. Procaccia, Phys. Rev. E, **52**, 3858 (1995).
[11] V.S. L'vov and I. Procaccia, Phys. Rev Lett. **76**, 2896 (1996).
[12] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **52**, 4924 (1995).
[13] E.S.C. Ching, Phys. Rev. Lett. **70**, 283 (1993).
[14] S.B. Pope and E.S.C. Ching, Phys. Fluids A **5**, 1529 (1993).
[15] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3608 (1995).